TenSyGrid

Tensors for System Analysis of Converter-dominated Power Grids

D 3.1 Report on Identified Non-linearities and Saturations

by eRoots

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About TenSyGrid

The demand for the power grid in Europe is undergoing profound changes due to an increasing number of decentralized feed-in points and the fluctuating supply from renewable energies. This complexity in interactions between power grid components poses a challenge for maintaining system stability. To address this, the European project TenSyGrid is developing a toolbox for direct stability assessment using multilinear models to capture the complex dynamics of power grid components. The objective is to support grid operators in assessing large power grids primarily powered by renewable energy. The toolbox will be compatible with existing commercial software packages to facilitate integration into current workflows.

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Publishable Summary

The power grid is considered to be the largest industrial system in mankind. As such, since the arrival of computer simulators, power systems have been extensively analyzed to ensure their robustness, efficiency and security of supply. In this context, dynamic analysis are fundamental to understand power system stability and transient behavior during disturbances. Real-world components, such as generators, loads, power lines, transformers, converters based on power electronics, among others, exhibit non-linear behaviors that must be accurately modeled to ensure simulation fidelity. While incorporating these non-linearities increases complexity, they are essential for capturing realistic dynamic behavior.

This document provides a complete set of the most common types of nonlinearities found in dynamic models, including saturation limits, squared and trigonometric terms, hysteresis, empirical relationships, discrete switching behaviors, and more. These non-linear elements regularly appear throughout the system. Hence, the goal of this deliverable is to provide a comprehensive overview of the non-linear building blocks that compose power system components, so that they can be later modelled in a multilinear format.

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1 Introduction

Power system dynamic analysis is crucial for assessing system stability and designing control strategies under disturbances. Accurate dynamic modeling of components like generators, loads, transmission lines, and control systems is critical to ensure simulations are closely matching with reality. However, real-world power system components exhibit numerous non-linear behaviors. Incorporating these non-linearities into models increases their fidelity but also significantly complicates analysis, often precluding closed-form analytical solutions and necessitating numerical simulation.

This document details common non-linearities encountered in power system dynamic models, identifying the components where they typically arise and providing the relevant mathematical formulations. Understanding these non-linearities is essential for interpreting simulation results and appreciating the complexities of power system behavior. In this spirit, the next deliverables will depart from these non-linearities and build equivalent multilinear models.

2 Differential and Integral Equations

2.1 Mathematical Description

Differential equations are defined such as $\frac{dx}{dt}$ (first derivative), whereas integral equations take the form of $\int_0^t x(\tau)d\tau$. Other derivative and integral terms can be composed based on these two basic operators.

Often, control systems and power system dynamics are represented through transfer functions. These transfer functions indicate the relationship between an output and an input. They are conventionally defined as $G(s) = \frac{Y(s)}{U(s)}$, where s is the complex Laplace variable, $Y(s)$ represents the output, and $U(s)$ symbolizes the input. Transfer functions with the Laplacian domain are particularly useful as they capture Linear Time-Invariant (LTI) dynamics in the frequency domain. In the time domain, they would correspond to a set of ordinary differential equations.

It is worth noting that transfer functions, and derivatives or integrals for that matter, are essentially linear operators. However, differential and algebraic equations can involve non-linear terms (e.g. a squared term, a sine, a cosine, etc.), hence the equations themselves are non-linear.

2.2 Occurrence in Power Systems

- **Controllers:** Controllers are used to regulate the output of a system and are typically represented as transfer functions. While they can take various forms, proportional-integral controllers are particularly common, and usually modelled in the s domain such as:

  $$G(s) = K_p + \frac{K_i}{s} \tag{2.1}$$

  where K_p is the proportional gain and K_i is the integral gain. Derivative terms are rarely used in power systems controls and thus excluded from the model.

- **Synchronous Generator Swing Equation (Second Order):** Describes the dynamics of the rotor angle (δ) and speed (ω), related to the power balance between the generator and the grid. The second order complete expression can be split into two first order equations:

$$\frac{d\delta}{dt} = \omega_b(\omega - \omega_s) \tag{2.2}$$

$$\frac{d\omega}{dt} = \frac{1}{2H}(P_m - P_e - D(\omega - \omega_s)) \tag{2.3}$$

  where H is the inertia constant, ω_b is base angular frequency, ω is rotor angular frequency (p.u.), ω_s is synchronous angular frequency (typically 1 p.u.), P_m is mechanical power, P_e is electrical power (often further involving $\sin(\delta)$ due to the grid's inductive nature), and D is the damping coefficient. The reader is invited to refer to [1] for a more detailed description of the swing equation.

- **Generator Flux Dynamics (First Order):** It is first common practice to refer the generator equations to its rotor reference frame (dq frame). Then, the relationship between the flux linkage ψ_d and ψ_q, current i_d and i_q, and voltage v_d and v_q is given by [2]:

$$\frac{d\psi_d}{dt} = v_d - R_s i_d + \omega\psi_q \tag{2.4}$$

$$\frac{d\psi_q}{dt} = v_q - R_s i_q - \omega\psi_d \tag{2.5}$$

  where the right-hand sides of the equations contain multilinear terms.

- **Control Filters:** Filters are used to remove undesired frequency components from a signal. For instance, low-pass filters remove high-frequency components from a signal, high-pass filters remove low-frequency components, and band-pass filters remove both high and low-frequency components. They are typically represented as transfer functions in the s domain. A low-pass filter transfer function, it its generic format, becomes:

$$G(s) = \frac{1}{\tau s + 1} \tag{2.6}$$

  where τ is the time constant, and its equivalent time domain expression is:

$$\frac{d}{dt}y(t) + \frac{1}{\tau}y(t) = \frac{1}{\tau}u(t) \tag{2.7}$$

- **Lead-Lag Compensators:** Power systems components can also integrate lead-lag compensators, which, as the name would indicate, are a combination of a low-pass and a high-pass filter. The transfer function of a lead-lag compensator is given by:

$$G(s) = K\frac{\tau_1 s + 1}{\tau_2 s + 1} \tag{2.8}$$

where K is a gain, τ_1 is the lead time constant, and τ_2 is the lag time constant. Power System Stabilizers (PSS) are usually implemented through lead-lag compensators as the basic building blocks [3]. Equivalently, the time domain expression of a lead-lag compensator is:

$$\tau_2\frac{d}{dt}y(t) + y(t) = K\tau_1\frac{d}{dt}u(t) + Ku(t) \tag{2.9}$$

2.3 Implications

Transfer functions are the standard way to represent linear dynamic components, especially controllers. They should be seen as the fundamental operators that one can expect to find in a power system model. It is also worth noting that despite the integral term, power system simulators tend to use a set of differential and algebraic equations to represent the dynamics of the system [4]. If the derivative and integral operators do not always return linear functions (in the sense of $f(x) \neq ax+b$, being f an output of the operators), it is because the operators are applied to non-linear functions to begin with. In the following work we explore the non-linearities that can arise in both differential and algebraic equations inside power systems.

3 Saturation Limits

3.1 Mathematical Description

Physical components have limits that have to be respected. Saturation blocks represent imposed limits on variables. Saturations can be due to physical limits inherent to the device (e.g. a transformer magnetic core saturation) or due to control actions (e.g. a power electronic converter current limitation). Often they serve the purpose of protecting the device from damage. Mathematically, saturations can be modeled as a piecewise-linear function:

$$y = \text{sat}(x) = \begin{cases} y_{\text{max}} & \text{if } x_{\text{max}} \leq x \\ k \cdot x & \text{if } x_{\text{min}} < x < x_{\text{max}} \\ y_{\text{min}} & \text{if } x \leq x_{\text{min}} \end{cases} \tag{3.1}$$

where typically $k = 1$ if x is the direct input to the limiter, x_{min} and x_{max} are the minimum and maximum input thresholds, and y_{min} and y_{max} are the minimum and maximum output limits. A more compact form, assuming $k = 1$ and thresholds aligned with output limits, uses min/max functions:

$$y = \min(y_{\text{max}}, \max(y_{\text{min}}, x)) \tag{3.2}$$

The saturation expression is especially tricky to handle given that in the intermediate region, being x between x_{min} and x_{max}, the output y is a linear function of x. However, the function is not linear in the whole domain, nor continuously differentiable.

3.2 Occurrence in Power Systems

Saturation limits appear across various control components in power systems, often to represent the physical limitations of actuators or ensure system protection, as previously discussed.

- **Synchronous Generator Excitation Systems:** One of the most common applications. The exciter output voltage (E_{fd} or V_R) is limited by the capabilities of the excitation hardware.

- **Equation Example (IEEE AC1A):** The voltage V_R feeding the main exciter field is limited, in a compact manner [5]:

$$V_R = \min(V_{R,\text{max}}, \max(V_{R,\text{min}}, K_A(V_{\text{ref}} - V_{\text{comp}} - V_{\text{stab}} - \frac{K_F \cdot E_{fd}}{sT_F}))) \tag{3.3}$$

  where $V_{R,\text{max}}$ and $V_{R,\text{min}}$ represent the regulator limits, K_A is the gain, V_{ref} is the reference voltage, V_{comp} is the compensation voltage, V_{stab} is the stabilizer voltage, K_F is the feedback gain, and T_F is the feedback time constant

  Additionally, the field voltage E_{fd} may be subject to ceiling limits influenced by exciter saturation:

$$E_{fd} = \min(V_E \cdot S_E(V_E)_{\text{max}}, \max(V_E \cdot S_E(V_E)_{\text{min}}, V_E)) \tag{3.4}$$

  where $S_E(V_E)$ is the exciter saturation function.

- **Power System Stabilizers (PSS):** They are used to prevent synchronous machines from going unstable due to electromechanical oscillations, provoked for instance by switching events. The PSS sends a signal related to the generator's motion, which is then added to the reference voltage of the Automatic Voltage Regulator (AVR).

  - **Equation Example (IEEE PSS2B):** The input signal is passed through to transfer functions $G_1(s)$ and $G_2(s)$, built with filters and lead-lag compensators and then multiplied by a gain K_{PSS} [6]:

$$V_s(s) = K_{\text{PSS}}(G_1(s) \cdot \text{PSS}_1\text{Input} + G_2(s) \cdot \text{PSS}_2\text{Input}) \tag{3.5}$$

  The output is then limited by the PSS limits:

$$V_s = \min(V_{\text{STMAX}}, \max(V_{\text{STMIN}}, V_s(s))) \tag{3.6}$$

  where V_{STMAX} and V_{STMIN} are the maximum and minimum output limits.

- **Governor Control Systems:** Governor systems are in charge of controlling the power output of generators. Mechanical actuators, like valve or gate positions, have imposed limits that must be respected given the physical constraints of the device.

- **Equation Example (Hydro Turbine Governor):** In simple terms, a hydro turbine or any other mechanical actuator has a limited range of motion. This is translated into a limit on the power output of the generator:

$$P = \min(P_{\text{MAX}}, \max(P_{\text{MIN}}, \text{Control Signal})) \tag{3.7}$$

where P_{MAX} and P_{MIN} are the maximum and minimum power output limits.

- **Power Converters:** Saturation limits are ubiquitous in control of power electronic converters (e.g., VSCs, MMCs). These devices have to face the limits of the physical components, like semiconductors, that compose them. In their control structures, limiters are applied to:

  - *Current Limits:* The output of the current controller is limited to avoid overcurrent in the semiconductors (IGBTs usually), and thus, not damage the converter. It is also typical to establish a rule to prioritize reactive over active current, or vice versa.

  - *Modulation Index:* To ensure PWM duty cycles remain between 0 and 1. The duty cycle is related to the ratio between the DC and AC voltages.

  - *Voltage Limits:* The converter's output voltage is limited to avoid issues in relation to overvoltage (which could in turn condition the insulation and filter sizing).

  - **Example:** The saturations found in power converters are not essentially different, in mathematical terms, as the ones found in the previous examples. However, for a simple current saturation strategy, the expression is as follows:

$$i_{q,s}^2 = \min(i_{\text{max}}^2, i_q^2) \tag{3.8}$$

It is worth noting that the current can take both positive and negative values, so some attention has to be paid to the sign of the original. Additionally, the d component of the current could be calculated as $i_d = \sqrt{i_{\text{max}}^2 - i_{q,s}^2}$.

Moreover, a note can be made about anti windup strategies. When saturation is not properly applied in systems with integral control actions, it can lead to a phenomenon known as wind-up, where the integrator continues to accumulate error even though the actuator is no longer responding. This causes delayed recovery and potential instability when the system returns to the linear operating range. Converter controllers are no exception to this rule, and thus, anti-windup strategies are often applied. Constructively, they imply the use of a saturation block that is able to detect when the saturation is active, and then stop the integration (with more or less sophistication).

3.3 Implications

In short, saturation introduces nonlinearity into otherwise linear control loops, which can lead to piecewise system behavior and even discontinuities in the state trajectories. This complicates both analysis and design, as linear control theory may no longer be directly applicable near the limits.

Saturation also affects system stability margins. Operating near or within the saturation region can reduce damping or gain margin, making the system more sensitive to disturbances. Careful tuning and anti-windup strategies are often required to mitigate these effects.

From a simulation perspective, saturation functions can introduce stiffness and convergence challenges, particularly in stiff or real-time simulations. Discontinuous or sharp saturation logic may require careful numerical handling to ensure reliable simulation results. Otherwise, we may face divergent solutions or accumulate significant errors.

4 Squared Terms

4.1 Mathematical Description

Non-linearities in power systems frequently arise from relationships where one variable depends quadratically on another. A general expression for this kind of non-linearity is:

$$y = k \cdot x^2 \tag{4.1}$$

where k is a constant that characterizes the system or component behavior. These relationships introduce non-linear terms into system equations, which must be handled with iterative solution techniques such as Newton-Raphson. Note that the term x^2 can be replaced by any other power, such as x^3, x^4, yet the same principles apply. Also, products of the form $x \cdot y$ are not an issue a priori, as they already follow a multilinear format.

4.2 Occurrence in Power Systems

Several components and phenomena in power systems involve squared terms due to their physical or operational characteristics:

- **Constant Impedance Loads:** Loads are conventionally modelled with a ZIP model, that is, a combination of constant impedance, current, and power loads [7]. Thus, a component of these static load models is the constant impedance model, where both active and reactive power consumption are proportional to the square of the voltage magnitude. This occurs because the power drawn by a purely resistive or inductive load follows Ohm's Law, yielding:

$$P_L = P_0 \left(\frac{V}{V_0}\right)^2 = \frac{V^2}{R} = G \cdot V^2 \tag{4.2}$$

$$Q_L = Q_0 \left(\frac{V}{V_0}\right)^2 = \frac{V^2}{X} = B \cdot V^2 \tag{4.3}$$

where P_0, Q_0 are the nominal active and reactive powers at the nominal voltage V_0, V is the actual bus voltage magnitude, R, X are the equivalent resistance and reactance of the load, and $G = 1/R$ and $B = 1/X$ are the equivalent conductance and susceptance, respectively. In its most general form, the load model could adopt other exponents; it has been here particularized for order 2.

- Transmission Line Losses: Power losses in transmission lines arise from resistive and reactive elements, which are proportional to the square of the current magnitude. These losses are expressed as:

$$P_{\text{loss}} = I^2 R \tag{4.4}$$

$$Q_{\text{loss}} = I^2 X \tag{4.5}$$

where I is the current magnitude, R is the resistance, and X is the reactance of the line. It is worth noting that the loss terms are typically computed at the end, once the power flow solution has been obtained.

- Power Balance: The power flow equations are a good example of a system of non-linear equations, involving, among others, squared terms. The complex power balance at a bus, relating it to the admittance matrix $\underline{\boldsymbol{Y}}$ and the bus voltage vector $\underline{\boldsymbol{V}}$, is given by:

$$\underline{\boldsymbol{S}}_{\text{bus}} = \underline{\boldsymbol{V}}(\underline{\boldsymbol{Y}} \cdot \underline{\boldsymbol{V}})^* \tag{4.6}$$

where $\underline{\boldsymbol{S}}_{\text{bus}}$ is the complex power vector at the bus, which would then be split into real and reactive powers. The avid reader will have noticed that if the admittance matrix has a non-zero diagonal, products of squared voltages are present. Moreover, the power flow equations also exhibit considerable multinearity, that is, there are multilinear terms for every pair of voltages such as $\underline{V}_i \cdot \underline{V}_j^*$, being $i \neq j$.

- Induction Motor Torque Characteristics: In dynamic modelling of induction motors, the electromagnetic torque T typically depends non-linearly on the slip s_l (relative speed difference between the rotor and the synchronous speed). A typical expression is:

$$T = \frac{3}{\omega_s} \cdot \frac{E^2 \cdot \frac{R}{s_l}}{(\frac{R}{s_l})^2 + X^2}, \tag{4.7}$$

where E is the terminal voltage, R is the equivalent resistance, X is the equivalent reactance, and ω_s is the synchronous speed. The quadratic term reflects non-linear behavior in torque development, especially relevant in stability and motor-starting studies. Note that the slip s_l is defined as $s_l = (\omega_s - \omega_r)/\omega_s$, where ω_r is the rotor speed. Also, the term

E^2 is present, which is a squared voltage, and there are divisions of variables (non-linear by definition). However, their multilinearity equivalence is relatively straighforward.

4.3 Implications

As it has been shown, square terms are often present in power system models, and they can have significant implications on the system's behavior, both in the steady-state and dynamic domains. For instance, the power flow equations are a good example of a system of non-linear equations, involving, apart from other non-linearities, squared terms. Likewise, induction motors also exhibit non-linear behavior with squared operations. Although it has not been particularly detailed here, control systems such as the ones in power electronic converters can also involve the presence of squared terms, without loss of generality. In short, products of squared terms are one of the most common sources of non-linearities in power systems.

5 Spatiotemporal Partial Differential Equations

5.1 Mathematical Description

Equations involving derivatives with respect to both time (t) and one or more spatial dimensions (e.g., distance x along a line): $\frac{\partial u}{\partial t} = F(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots)$. These expressions can come from the Telegrapher's equations in the power systems realm, and are especially tricky to handle, as they are not linear and require special numerical techniques to solve.

5.2 Occurrence in Power Systems

- **Electromagnetic Transients (EMT) on Transmission Lines:** The Telegrapher's equations accurately model voltage $v(x,t)$ and current $i(x,t)$ wave propagation. They are a set of two coupled partial differential equations (PDE).
  - Equations: For a line with distributed resistance R', inductance L', conductance G', and capacitance C' per unit length [8]:

$$\frac{\partial v(x,t)}{\partial x} = -R' i(x,t) - L' \frac{\partial i(x,t)}{\partial t} \tag{5.1}$$

$$\frac{\partial i(x,t)}{\partial x} = -G' v(x,t) - C' \frac{\partial v(x,t)}{\partial t} \tag{5.2}$$

    These are linear PDEs, but non-linearities can arise from electromagnetic effects (voltage-dependent G' or C') or line arresters modelled with non-linear V-I characteristics. Note that, to the best of the author's knowledge, this is the only occurrence in power systems where the derivatives with respect of the position appear. Additionally, there is no general method to solve these equations analytically, hence they are solved using numerical techniques (see Dommel's algorithm [9]).

- **Distributed Parameter Models of Large Generators:** Highly specialized models might could consider spatial variations within the windings, akin to a transmission line. However, given the relatively minor length of these windings, compared to a transmission line, the spatial variations are rarely considered.

5.3 Bergeron Line Model

The Bergeron model is a simplified transmission line EMT model used in power system transient analysis, particularly for simulating voltage and current transients caused by sudden events, which deserves special attention. Some variables in the equations are composed with a lag function with time lag τ. This lag time is the wave travel time (or propagation delay) along the transmission line. These equations are a direct application of the EMT Transmission lines [10]:

$$V_s(t) - Z_c I_s(t) = V_r(t-\tau) - Z_c I_r(t-\tau) \tag{5.3}$$

$$V_r(t) - Z_c I_r(t) = V_s(t-\tau) - Z_c I_s(t-\tau) \tag{5.4}$$

where $V_s(t)$ and $I_s(t)$ are the voltage and current at the sending end at time t, $V_r(t)$ and $I_r(t)$ are the voltage and current at the receiving end at time t, Z_c is the characteristic impedance of the transmission line. For a lossless line, it is given by $Z_c = \sqrt{L'/C'}$, where L' and C' are the per-unit-length inductance and capacitance, respectively, and τ is the travel time of a wave along the line. For a lossless line, it is given by $\tau = d\sqrt{L'C'}$, where d is the length of the line.

The Bergeron model could be seen as a special case of the Telegrapher's equations, where the spatial derivatives are not present thanks to the discretization of the line in multiple sections, and where a lag is introduced to account for the wave travel time. Figure 5.1 shows the equivalent circuit section of the Bergeron model.

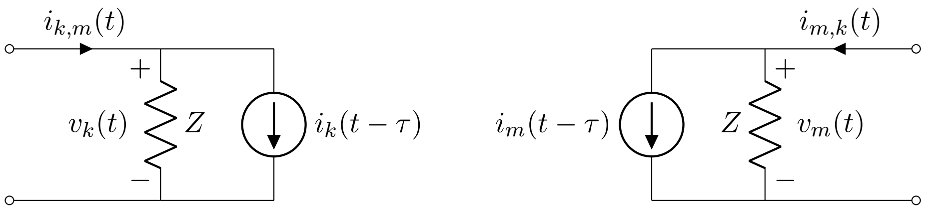

Figure 5.1: Bergeron Model Time Domain Interface.

5.4 Implications

Spatiotemporal derivatives are a different set of derivatives than the ones traditionally found in power systems machinery and control systems. Despite this, they are extensively known in the literature, and much needed to accurately represent the behavior of power lines. By discretizing the line in multiple sections, the Bergeron model is able to represent the behavior of the line with a good accuracy, while still being a relatively simple model. It is to be seen if TenSyGrid aims to discretize the equations and operate on them, or rather, depart from the Telegrapher's equations in their most pure form, and try to assess the stability from them.

6 Trigonometric Terms

6.1 Mathematical Description

Often, power systems models deal with trigonometric functions to represent rotations or the real and imaginary projections of phasors. Functions like $\sin(x)$ and $\cos(x)$ are recurrently found. Additionally, signals such as $v = \sin(\omega t)$ and harmonics of that signal can also appear. Harmonics would have the mathematical expression $\sin(n\omega t)$ with $n \in \mathbb{N}$.

6.2 Occurrence in Power Systems

- **Power Transfer Equations:** They establish the fundamental relationship between power and voltage magnitudes and angles. The power flow problem consists of solving them. They can be expressed in rectangular form or polar form, being the latter the most common [11]:

$$P_i = \sum_{j=1}^{N} |V_i||V_j|(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j)) \tag{6.1}$$

$$Q_i = \sum_{j=1}^{N} |V_i||V_j|(G_{ij}\sin(\theta_i - \theta_j) - B_{ij}\cos(\theta_i - \theta_j)) \tag{6.2}$$

  where $V_k = |V_k|\angle\theta_k$ is the voltage phasor at bus k, and $Y_{ij} = G_{ij} + jB_{ij}$ is the ij-th element of the network admittance matrix.

- **Synchronous Generator Electrical Torque/Power:** The air-gap torque (T_e) or power (P_e) depends sinusoidally on the rotor angle (δ) relative to the terminal voltage or infinite bus angle. These trigonometric function appear when expressing the generator's internal variables in the dq-reference frames. Such a change of reference is necessary to model part of the synchronous generators behavior.

  – Classical Model: Connected to an infinite bus $V\angle 0$, assuming a perfectly inductive behavior:

$$P_e = \frac{E'V}{X}\sin(\delta) \tag{6.3}$$

Where E' is internal voltage magnitude, V is infinite bus voltage magnitude, X is the machine's equivalent reactance, and δ is rotor angle relative to the infinite bus. This simplified expression is often used to analyze the transient stability of the system [12].

– Salient Pole - Simplified: Including reluctance torque, relative to terminal voltage $V_t\angle\theta_t$, as similarly found in [13]:

$$P_e = \frac{E'_q V_t}{X'_d}\sin(\delta-\theta_t) + V_t^2\frac{X'_d - X'_q}{2X'_d X'_q}\sin(2(\delta-\theta_t)) \tag{6.4}$$

where E'_q is q-axis internal voltage, V_t is terminal voltage magnitude, X'_d, X'_q are d- and q-axis transient reactances.

- **Park Transformation:** Found within converter controls, it is used extensively both in grid-following and grid-forming converters to move from the abc-reference frame to the dq-reference frame. The core error signal often involves trigonometric functions of the angle difference. In its most complete format, including the Clarke transformation in it, the transformation is given by [14]:

$$\begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} \cos(\theta) & \cos(\theta-\frac{2\pi}{3}) & \cos(\theta+\frac{2\pi}{3}) \\ -\sin(\theta) & -\sin(\theta-\frac{2\pi}{3}) & -\sin(\theta+\frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix} \tag{6.5}$$

where θ is the angle between the d-axis and the phase-a axis (tracked with a phase-locked loop in grid-following converters), and x_a, x_b, x_c are the components of the vector x in the abc-reference frame. Note that factors and different dq versus qd ordering are present in the literature, although the core idea is the same.

6.3 Implications

Trigonometric functions are also one of the most common sources of nonlinearity in power systems, found in many models, including conventional machines, power lines and converters. The complexity is in how to multilinearize expressions such as $\sin(x)$ and also $\sin(x + a)$ where a is a constant.

7 Empirical Non-Linear Relationships

7.1 Mathematical Description

These relationships are derived from experimental observations rather than fundamental physical laws, although they can be approximated by fitted analytical functions. They are typically represented using:

- **Look-Up Tables (LUTs):** Discrete mappings from input to output values, often combined with interpolation methods. Expressed as $y = \text{LUT}(x)$.
- **Piecewise Functions:** Multiple analytical expressions defined over distinct intervals of the input variable. Their form could resemble that of (3.1), with potentially more conditionals.
- **Fitted Analytical Functions:** Parametric models (e.g., polynomials, exponentials) calibrated to match empirical data.

7.2 Occurrence in Power Systems

- **Magnetic Saturation:** As found in generators and transformers, the non-linear relationship between current (I) and the magnetic flux (or internal voltage E) arises from magnetic core saturation. This behavior is typically characterized using empirical data from the Open Circuit Characteristic (OCC) curve of the machine.
  - **Example:** A saturation factor S_E modifies the air-gap voltage contribution, capturing the increased excitation current required due to saturation. S_E is a non-linear function of the internal voltage E or terminal voltage V_t, and is often represented as:

$$S_E(E) = A_{\text{sat}} \exp(B_{\text{sat}} \cdot E) \quad \text{(Exponential fit model)} \tag{7.1}$$

where the exponential is a growing exponential, that is, $B_{\text{sat}} > 0$. Alternatively, it may be defined through a look-up table derived from OCC data. In excitation system models, the saturation effect is commonly modeled as an additive correction to the linear air-gap voltage. More complex models include the usage of exponential Fourier approximations [15], yet this is left out of the scope of this manuscripts.

- **Load Modeling:** The voltage and frequency dependence of aggregated loads may not conform to simple analytic expressions and are frequently captured through empirical models, such as the ZIP model. The load ZIP model has been presented before, but not with the dependency on frequency.

  - **Example:** The combined ZIP load model expresses active and reactive power as:

$$P_L = P_0 \left(a_z \left(\frac{V}{V_0} \right)^2 + a_i \left(\frac{V}{V_0} \right) + a_p \right) (1 + k_{pf} \Delta f) \tag{7.2}$$

$$Q_L = Q_0 \left(b_z \left(\frac{V}{V_0} \right)^2 + b_i \left(\frac{V}{V_0} \right) + b_p \right) (1 + k_{qf} \Delta f) \tag{7.3}$$

    where $a_z + a_i + a_p = 1$, $b_z + b_i + b_p = 1$, and Δf represents the frequency deviation. The coefficients a_k, b_k, k_{pf}, and k_{qf} are typically obtained through empirical fitting. The positive side of using this model is that it contains linear operations apart from the squared voltage term, which has been previously discussed.

- **PV Panel Characteristics:** The current-voltage (I-V) behaviour of photovoltaic panels under specific irradiance and temperature conditions is inherently non-linear and usually modeled using empirical data or semi-empirical diode-based equations [16]. More on that in the following section.

7.3 Implications

Empirical models are essential when physics-based formulations are either unavailable or too complex to be practical. They allow accurate representation of real-world behaviour but rely heavily on the quality and resolution of the underlying data. In particular, care must be taken when using piecewise functions to avoid introducing discontinuities, and the choice of interpolation or fitting technique can significantly impact model fidelity. Nonetheless, the usage of empirical models do not introduce novel non-linearities, but rather a different way of representing known non-linearities.

8 Exponentials

8.1 Mathematical Description

Exponential functions describe relationships where the rate of change of a variable is proportional to its current value. They are generally expressed as:

$$y = a \cdot \exp(b \cdot x) \quad \text{or} \quad y = a \cdot (\exp(bx) - 1)$$

where a and b are fitting parameters that control the scaling and growth rate of the function. These forms are particularly useful for capturing behaviours that involve sharp nonlinear growth or decay. From our experience, convergence of numerical solvers can be hard to achieve when using exponential functions, as they can introduce stiffness to the system.

8.2 Occurrence in Power Systems

Exponential functions appear frequently in power system modeling, especially in contexts where physical phenomena exhibit strong non-linearities over certain operating ranges. Notable applications include:

- **Generator Saturation Models:** As previously discussed in (7.1), the magnetic saturation characteristic of synchronous machines and transformersis commonly approximated using exponential functions. This formulation captures the rapid increase in excitation current required as the machine approaches core saturation.

- **Power Electronics (Diode/Thyristor Models):** Semiconductor devices exhibit exponential voltage-current relationships, as indicated by the Shockley diode equation. This is especially relevant in detailed electromagnetic transient (EMT) models of converters, rectifiers, and switching devices. Notoriously, the Shockley diode equation is written as [17]:

$$I_D = I_S \left(\exp\left(\frac{V_D}{nV_T} \right) - 1 \right) \tag{8.1}$$

where I_D is the diode current, V_D is the diode voltage, I_S is the reverse saturation current, n is the ideality factor (typically between 1 and 2), and V_T is the thermal voltage. This equation captures the sharply non-linear conduction behaviour of junction-based semiconductor devices.

It is not uncommon to model diodes and similar semiconductors in a piecewise format to avoid confronting the heavily varying derivative of the exponential function, especially in SPICE-like software. Also, the exponential function is used in the modeling of photovoltaic panels.

8.3 Implications

Exponential functions are critical for accurately representing physical processes characterized by sharp non-linearities, such as magnetic saturation in machines and current-voltage characteristics in semiconductors. Their inclusion enhances the realism of simulation models, especially in transient and dynamic studies. However, care must be taken when incorporating exponential terms in numerical simulations, as their steep gradients can introduce stiffness and convergence challenges in time-domain solvers. It will have to be discussed if this level of detail is required in TenSyGrid.

9 Hysteresis

9.1 Mathematical Description

Hysteresis refers to a system behaviour where the output $y(t)$ depends not only on the current input $x(t)$, but also on the history of the input—particularly the direction of change. This can be expressed as:

$$y(t) = F(x(t), \text{history}(x))$$

As a result, the system may produce different output values for the same input, depending on whether the input is increasing or decreasing. This characteristic leads to the formation of a closed loop in the x-y plane, known as a hysteresis loop.

9.2 Occurrence in Power Systems

- **Transformer Core Magnetization (B–H Curve):** The most prominent example of hysteresis in power systems is found in the magnetization of transformer cores. The relationship between magnetic flux density (B) and magnetic field intensity (H) displays hysteresis due to the behaviour of magnetic domains in ferromagnetic materials. Accurate representation of this phenomenon is essential for modelling:
  - *Inrush Currents:* High transient currents that occur when energizing a transformer. Critical to model for the design of protective relays and tuning of circuit breakers.
  - *Ferroresonance:* A complex and potentially damaging resonant condition involving nonlinear inductance and system capacitance.
  - *Harmonics:* Nonlinear magnetization can introduce harmonic distortion into the current waveform, amplifying/dampening some harmonics based on their frequency.

  - *Equations:* Detailed hysteresis models—such as the Preisach and Jiles Atherton models—use state-dependent differential or integral equations to describe magnetic behavior [18]. Simpler implementations may rely on look-up tables or empirical approximations of the hysteresis loop. In many stability or phasor-domain studies, hysteresis is often neglected, and a single-valued saturation curve is used instead. For EMT simulations, introducing conditionals as in the saturation block is a common approach.

- **Relays (e.g., Thermostats, Certain Protection Relays):** Many relays incorporate hysteresis in their logic, switching ON at one threshold and OFF at a different one. This prevents frequent toggling due to small fluctuations around a threshold and improves robustness in control logic.

Figure 9.1 shows an example of a hysteresis model block to illustrate the dependency on the history of the input.

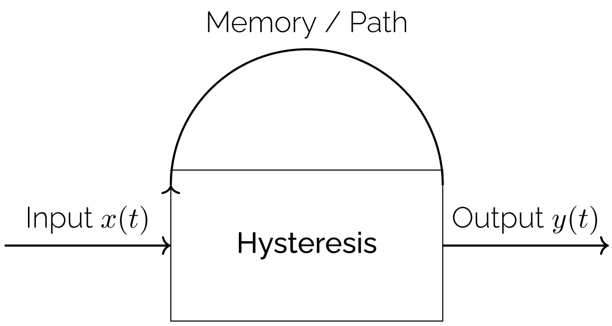

Figure 9.1: Example of a hysteresis model block.

9.3 Implications

Hysteresis introduces memory and path dependency into system behaviour, making the current state dependent on past inputs. While this promotes modelling realism, it also increases the complexity of simulation, especially in time-domain studies. Initializing the models at the correct initial conditions is crucial for accurate results. As a result, detailed hysteresis models are typically used only in electromagnetic transient (EMT) simulations or specific applications such as ferroresonance analysis.

10 Absolute Value

10.1 Mathematical Description

The absolute value function returns the non-negative magnitude of a real number, regardless of its sign. It is defined as:

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{10.1}$$

This piecewise-linear function is continuous but not differentiable at $x = 0$, which introduces modelling and numerical implications in systems that rely on smooth behaviour.

10.2 Occurrence in Power Systems

The absolute value function arises in various power system applications, particularly where the system must respond to the magnitude of a variable regardless of its direction:

- **Control System Deadbands:** Many control systems are designed with deadband logic, meaning the controller remains inactive when the input error remains within a certain tolerance. The control action is only triggered when the absolute value of the error exceeds the deadband threshold.
  - **Example:** Given an input error e, the controller output u can be expressed as:

$$u = \begin{cases} 0 & \text{if } |e| \leq \text{Deadband} \\ K \cdot (e - \text{sign}(e) \cdot \text{Deadband}) & \text{if } |e| > \text{Deadband} \end{cases} \tag{10.2}$$

    This ensures that small oscillations or noise around the setpoint do not cause unnecessary control actions, as explained in the hysteresis section.

- **Protection Functions:** Protective relays often operate based on signal magnitudes. Overcurrent protection, for instance, triggers when the absolute value of the measured current exceeds a preset limit. Similarly, distance protection functions rely on the magnitude of voltage and current phasors (e.g., $|v|, |i|$) to calculate apparent impedance and determine fault conditions [19].

One approach towards implementing the absolute value of a complex signal is to square the components and take the square root, such as for instance $S = \sqrt{P^2 + Q^2}$. This is equivalent to $S = |P + jQ|$. Opting for the squares and the root involves non-linear operations as well, by introducing i) the squares, and ii) the square root. It is to be seen which approach is more numerically stable.

10.3 Implications

The absolute value function introduces a non-smooth point at $x = 0$, where its derivative changes discontinuously from -1 to $+1$. This sharp transition can lead to challenges in numerical solvers, particularly in time-domain simulations involving control loops. In optimization and controller tuning, the non-differentiability at zero may affect convergence properties or necessitate the use of smooth approximations such as $y \approx \sqrt{x^2 + \epsilon}$ to ensure numerical stability. Despite its simplicity, the absolute value function is a fundamental and common non-linear block to deal with.

11 Discrete Switching

11.1 Mathematical Description

Discrete functions describe behaviours that exhibit abrupt transitions between distinct states, typically binary. A discrete event can be represented as a time-dependent binary function that activates at a specific time t_0, such as:

$$f(t) = \begin{cases} 1 & \text{if } t \geq t_0 \\ 0 & \text{if } t < t_0 \end{cases} \tag{11.1}$$

Logical switches are an extension of this concept, where the system maintains an internal state that evolves based on a logical rule. These can be formalised using a discrete state variable $b(t) \in \{0, 1\}$ that updates at each time step according to a user-defined transition function f:

$$b(t) \in \{0, 1\} \tag{11.2}$$

$$b(t+1) = f(b(t), x(t)) \tag{11.3}$$

where $x(t)$ is an input signal at time t. Such constructs are widely used in digital control, mode-dependent systems, and hybrid models that mix continuous dynamics with discrete logic.

11.2 Occurrence in Power Systems

Various non-continuous functions are employed in control and protection logic. These binary-valued functions introduce abrupt transitions in system behaviour based on thresholds:

- **Heaviside Function:** Symbolize a step change in the input and were first introduced by Oliver Heaviside [20]:

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{11.4}$$

This function is used to model switching behaviour that occurs at zero-crossings of the input. Heaviside functions are employed in control systems to for instance impose a reference step-change, e.g., an immediate and change in the reference active power setpoint.

- **Step Function at Threshold:** Similary, the Heaviside function can be generalised to arbitrary thresholds x_0, which can represent an electrical magnitude, or time:

$$h(x) = \begin{cases} 1 & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0 \end{cases} \tag{11.5}$$

  This generalizes the Heaviside function to arbitrary thresholds.

- **Bang Bang Controllers:** A bang-bang controller is a control mechanism that switches instantaneously between two extreme output values with no intermediate states. This form of control is particularly common in systems where a binary actuation is sufficient. Thermostats, limit controllers, and converter gating logic are good examples.

  Unlike smooth controllers, bang-bang logic often incorporates an internal discrete state that governs the switching logic. This results in a hybrid model combining continuous inputs and outputs with discrete internal dynamics. Let $u \in \mathbb{R}$ be the input signal, $y \in \mathbb{R}$ the output signal, and $b \in \{0, 1\}$ a binary state variable representing the current logical mode of the controller. The controller toggles between two output levels y_{min} and y_{max}, depending on whether u exceeds the upper or lower hysteresis thresholds u_{max} and u_{min}, respectively.

  The switching logic is typically defined by an update rule such as:

  - If $b(t) = 0$ and $u(t) > u_{\text{max}}$, then set $b(t + 1) = 1$
  - If $b(t) = 1$ and $u(t) < u_{\text{min}}$, then set $b(t + 1) = 0$

  This introduces hysteresis into the control action and avoids chattering caused by noise or small fluctuations around the threshold. The final control output is then given by:

$$y(t) = \begin{cases} y_{\text{max}} & \text{if } b(t) = 1 \\ y_{\text{min}} & \text{if } b(t) = 0 \end{cases} \tag{11.6}$$

  Such controllers are simple to implement and robust, ideal for systems where fast, reliable switching is required. However, the discontinuities they introduce must be treated carefully in numerical simulations.

11.3 Implications

Discrete switching elements introduce discontinuities and hybrid dynamics into power system models, which can significantly impact numerical simulation and analysis. The abrupt transitions between states can cause convergence issues in numerical solvers, especially when multiple switching events occur in close succession. The non-smooth nature of switching can also complicate the development of stability analysis given the fact that state space representations would heavily depend on the operating point.

12 Conclusion

Power system dynamic models are inherently non-linear due to the fundamental physics of AC power generation and transmission, physical limitations of equipment, and the behavior of control systems. The non-linearities discussed, including saturation, squared terms, trigonometric functions, empirical relationships, exponentials, hysteresis, absolute values, and the interactions involving dynamic elements represented by derivatives and transfer functions, are critical for capturing realistic system behavior.

Accurate modeling of these non-linearities is essential for the upcoming stability assessment with multilinear models and computational tools. The choice of which non-linearities to include and how accurately to model them often involves a trade-off between simulation fidelity and computational feasibility, depending on the specific phenomenon under investigation (e.g., transient stability, electromagnetic transients, voltage stability, small-signal stability). Hopefully, this first deliverable will be a useful resource for the work that follows.

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