Small-Signal Tools for Stability Analysis in Modern Power Systems

Part II – From Matrices to Tensors

TenSyGrid

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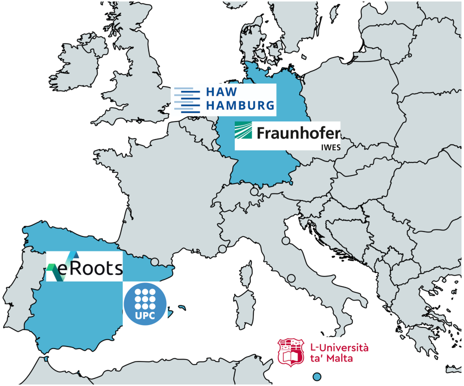
TENSYGRID

Overview

Tensors for System Analysis of Converter-dominated Power Grids (TenSyGrid) will develop innovative **stability analysis methods** to facilitate the integration of **100% renewable-**based energy systems while maintaining a **safe and stable** operation of the power system.

- Call: CETP CM2023-02 Energy system flexibility: renewables production, storage and system integration
- Start date: December 1st, 2024
- Duration: 36 months
- Budget: 1.5 M€
- EU Funding: 95%
- 5 Partners, 3 Countries



TENSYGRID

Specific Objectives

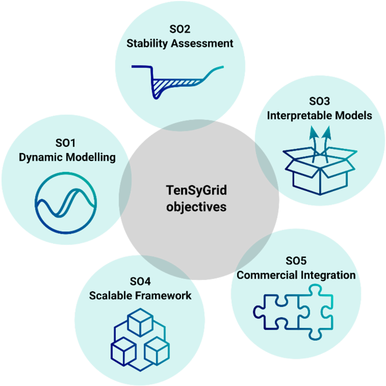
SO1 Obtaining a **modeling framework** for power systems with high penetration of power electronic devices that **captures the relevant nonlinear dynamics**.

SO2 Assist decision makers with **real-time power system stability assessments** under high penetration of renewable energy.

SO3 Overcome the **lack of interpretability of black-box models** from original equipment manufacturers hindering root-cause analysis.

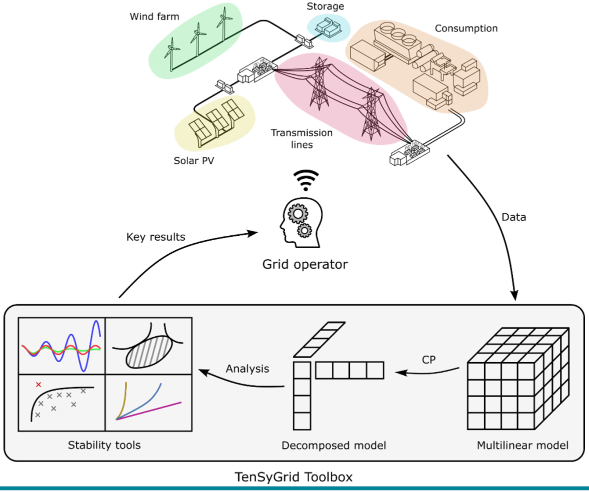
SO4 Obtaining a **modeling framework** that is easily **scalable** to extremely large power systems and easily **updatable** to address ongoing structural changes in the power system network.

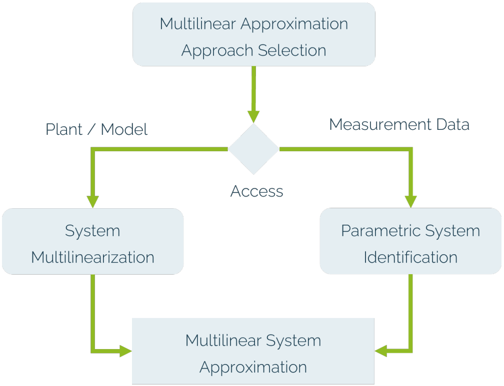
SO5 Developing tools that **are compatible with existing commercial software** packages and new methodologies that are easily accessible.



TENSYGRID

Use cases





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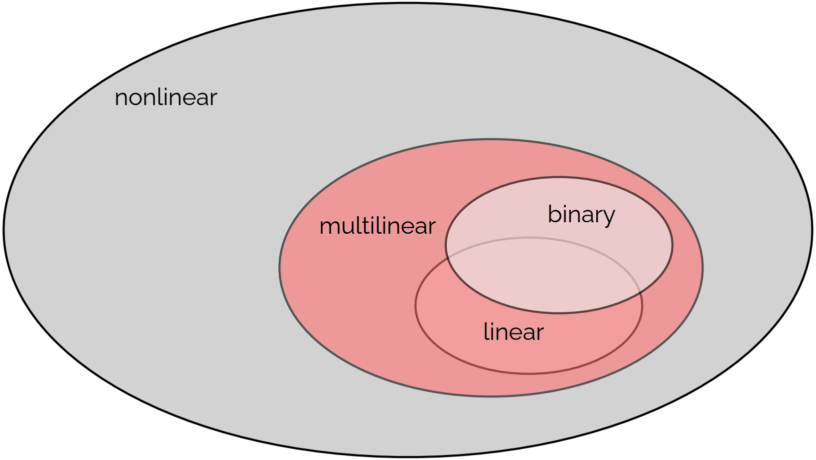
SMALL-SIGNAL POWER SYSTEM STABILITY ANALYSIS USING MULTILINEAR MODELS

Gerwald Lichtenberg (HAW Hamburg)

Multilinear modelling: why and how?

MODEL CLASSES



MULTILINEAR FUNCTIONS ARE POLYNOMIALS WITH EXPONENTS AT MOST 1

Vector of monomials

$$\mathbf{m}(x_1, x_2) = \begin{pmatrix} 1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1 x_2 \end{pmatrix}$$

$$\mathbf{m}(x_1, x_2, u) = \begin{pmatrix} 1 \\ u \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1 x_2 u \\ x_1 u \\ x_2 u \\ x_1 x_2 u \end{pmatrix}$$

EXPONENTIAL COMPLEXITY (2^n) WITH MODEL ORDER n, FOR EXAMPLE $n = 8$

$$
\begin{aligned}
&x_1x_2x_3x_4x_5x_6x_7x_8 + 8x_1x_2x_3x_4x_5x_6x_7 + 7x_1x_2x_3x_4x_5x_6x_8 + 6x_1x_2x_3x_4x_5x_7x_8 + 5x_1x_2x_3x_4x_6x_7x_8 \\
&+4x_1x_2x_3x_5x_6x_7x_8 + 3x_1x_2x_4x_5x_6x_7x_8 + 2x_1x_3x_4x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 + 56x_1x_2x_3x_4x_5x_6 \\
&+48x_1x_2x_3x_4x_5x_7 + 42x_1x_2x_3x_4x_5x_8 + 40x_1x_2x_3x_4x_6x_7 + 35x_1x_2x_3x_4x_6x_8 + 30x_1x_2x_3x_4x_7x_8 \\
&+32x_1x_2x_3x_5x_6x_7 + 28x_1x_2x_3x_5x_6x_8 + 24x_1x_2x_3x_5x_7x_8 + 20x_1x_2x_3x_6x_7x_8 + 24x_1x_2x_4x_5x_6x_7 \\
&+21x_1x_2x_4x_5x_6x_8 + 18x_1x_2x_4x_5x_7x_8 + 15x_1x_2x_4x_6x_7x_8 + 12x_1x_2x_5x_6x_7x_8 + 16x_1x_3x_4x_5x_6x_7 \\
&+14x_1x_3x_4x_5x_6x_8 + 12x_1x_3x_4x_5x_7x_8 + 10x_1x_3x_4x_6x_7x_8 + 8x_1x_3x_5x_6x_7x_8 + 6x_1x_4x_5x_6x_7x_8 \\
&+8x_2x_3x_4x_5x_6x_7 + 7x_2x_3x_4x_5x_6x_8 + 6x_2x_3x_4x_5x_7x_8 + 5x_2x_3x_4x_6x_7x_8 + 4x_2x_3x_5x_6x_7x_8 \\
&+3x_2x_4x_5x_6x_7x_8 + 2x_3x_4x_5x_6x_7x_8 + 336x_1x_2x_3x_4x_5 + 280x_1x_2x_3x_4x_6 + 240x_1x_2x_3x_4x_7 \\
&+210x_1x_2x_3x_4x_8 + 224x_1x_2x_3x_5x_6 + 192x_1x_2x_3x_5x_7 + 168x_1x_2x_3x_5x_8 + 160x_1x_2x_3x_6x_7 \\
&+140x_1x_2x_3x_6x_8 + 120x_1x_2x_3x_7x_8 + 168x_1x_2x_4x_5x_6 + 144x_1x_2x_4x_5x_7 + 126x_1x_2x_4x_5x_8 \\
&+120x_1x_2x_4x_6x_7 + 105x_1x_2x_4x_6x_8 + 90x_1x_2x_4x_7x_8 + 96x_1x_2x_5x_6x_7 + 84x_1x_2x_5x_6x_8 + 80x_1x_3x_4x_6x_7 \\
&+72x_1x_2x_5x_7x_8 + 60x_1x_2x_6x_7x_8 + 112x_1x_3x_4x_5x_6 + 96x_1x_3x_4x_5x_7 + 84x_1x_3x_4x_5x_8 + 48x_1x_4x_5x_6x_7 \\
&+70x_1x_3x_4x_6x_8 + 60x_1x_3x_4x_7x_8 + 64x_1x_3x_5x_6x_7 + 56x_1x_3x_5x_6x_8 + 48x_1x_3x_5x_7x_8 + 40x_1x_3x_6x_7x_8 \\
&+42x_1x_4x_5x_6x_8 + 36x_1x_4x_5x_7x_8 + 30x_1x_4x_6x_7x_8 + 24x_1x_5x_6x_7x_8 + 56x_2x_3x_4x_5x_6 + 48x_2x_3x_4x_5x_7 \\
&+42x_2x_3x_4x_5x_8 + 40x_2x_3x_4x_6x_7 + 35x_2x_3x_4x_6x_8 + 30x_2x_3x_4x_7x_8 + 32x_2x_3x_5x_6x_7 + 28x_2x_3x_5x_6x_8 \\
&+24x_2x_3x_5x_7x_8 + 20x_2x_3x_6x_7x_8 + 24x_2x_4x_5x_6x_7 + 21x_2x_4x_5x_6x_8 + 18x_2x_4x_5x_7x_8 + 15x_2x_4x_6x_7x_8 \\
&+12x_2x_5x_6x_7x_8 + 16x_3x_4x_5x_6x_7 + 14x_3x_4x_5x_6x_8 + 12x_3x_4x_5x_7x_8 + 10x_3x_4x_6x_7x_8 + 8x_3x_5x_6x_7x_8 \\
&+6x_4x_5x_6x_7x_8 + 1680x_1x_2x_3x_4 + 1344x_1x_2x_3x_5 + 1120x_1x_2x_3x_6 + 960x_1x_2x_3x_7 + 840x_1x_2x_3x_8 \\
&+1008x_1x_2x_4x_5 + 840x_1x_2x_4x_6 + 720x_1x_2x_4x_7 + 630x_1x_2x_4x_8 + 672x_1x_2x_5x_6 + 576x_1x_2x_5x_7 \\
&+504x_1x_2x_5x_8 + 480x_1x_2x_6x_7 + 420x_1x_2x_6x_8 + 360x_1x_2x_7x_8 + 672x_1x_3x_4x_5 + 560x_1x_3x_4x_6 \\
&+480x_1x_3x_4x_7 + 420x_1x_3x_4x_8 + 448x_1x_3x_5x_6 + 384x_1x_3x_5x_7 + 336x_1x_3x_5x_8 + 320x_1x_3x_6x_7 \\
&+280x_1x_3x_6x_8 + 240x_1x_3x_7x_8 + 336x_1x_4x_5x_6 + 384x_1x_3x_5x_7 + 336x_1x_3x_5x_8 + 320x_1x_3x_6x_7 \\
&+210x_1x_4x_6x_8 + 180x_1x_4x_7x_8 + 192x_1x_5x_6x_7 + 168x_1x_5x_6x_8 + 144x_1x_5x_7x_8 + 120x_1x_6x_7x_8 \\
&+336x_2x_3x_4x_5 + 280x_2x_3x_4x_6 + 240x_2x_3x_4x_7 + 210x_2x_3x_4x_8 + 224x_2x_3x_5x_6 + 192x_2x_3x_5x_7 \\
&+168x_2x_3x_5x_8 + 160x_2x_3x_6x_7 + 140x_2x_3x_6x_8 + 120x_2x_3x_7x_8 + 168x_2x_4x_5x_6 + 144x_2x_4x_5x_7 \\
&+126x_2x_4x_5x_8 + 120x_2x_4x_6x_7 + 105x_2x_4x_6x_8 + 90x_2x_4x_7x_8 + 96x_2x_5x_6x_7 + 84x_2x_5x_6x_8 \\
&+72x_2x_5x_7x_8 + 60x_2x_6x_7x_8 + 112x_3x_4x_5x_6 + 96x_3x_4x_5x_7 + 84x_3x_4x_5x_8 + 80x_3x_4x_6x_7 + 70x_3x_4x_6x_8 \\
&+60x_3x_4x_7x_8 + 64x_3x_5x_6x_7 + 56x_3x_5x_6x_8 + 48x_3x_5x_7x_8 + 40x_3x_6x_7x_8 + 48x_4x_5x_6x_7 + 42x_4x_5x_6x_8 \\
&+36x_4x_5x_7x_8 + 30x_4x_6x_7x_8 + 24x_5x_6x_7x_8 + 6720x_1x_2x_3 + 5040x_1x_2x_4 + 4032x_1x_2x_5 + 3360x_1x_2x_6 \\
&+2880x_1x_2x_7 + 2520x_1x_2x_8 + 3360x_1x_3x_4 + 2688x_1x_3x_5 + 2240x_1x_3x_6 + 1920x_1x_3x_7 + 1680x_1x_3x_8 \\
&+2016x_1x_4x_5 + 1680x_1x_4x_6 + 1440x_1x_4x_7 + 1260x_1x_4x_8 + 1344x_1x_5x_6 + 1152x_1x_5x_7 + 1008x_1x_5x_8 \\
&+960x_1x_6x_7 + 840x_1x_6x_8 + 720x_1x_7x_8 + 1680x_2x_3x_4 + 1344x_2x_3x_5 + 1120x_2x_3x_6 + 960x_2x_3x_7 \\
&+840x_2x_3x_8 + 1008x_2x_4x_5 + 840x_2x_4x_6 + 720x_2x_4x_7 + 630x_2x_4x_8 + 672x_2x_5x_6 + 576x_2x_5x_7 \\
&+504x_2x_5x_8 + 480x_2x_6x_7 + 420x_2x_6x_8 + 360x_2x_7x_8 + 672x_3x_4x_5 + 560x_3x_4x_6 + 480x_3x_4x_7 \\
&+420x_3x_4x_8 + 448x_3x_5x_6 + 384x_3x_5x_7 + 336x_3x_5x_8 + 320x_3x_6x_7 + 280x_3x_6x_8 + 240x_3x_7x_8 \\
&+336x_4x_5x_6 + 288x_4x_5x_7 + 252x_4x_5x_8 + 240x_4x_6x_7 + 210x_4x_6x_8 + 180x_4x_7x_8 + 192x_5x_6x_7 \\
&+168x_5x_6x_8 + 144x_5x_7x_8 + 120x_6x_7x_8 + 20160x_1x_2 + 13440x_1x_3 + 10080x_1x_4 + 8064x_1x_5 \\
&+6720x_1x_6 + 5760x_1x_7 + 5040x_1x_8 + 6720x_2x_3 + 5040x_2x_4 + 4032x_2x_5 + 3360x_2x_6 + 2880x_2x_7 \\
&+2520x_2x_8 + 3360x_3x_4 + 2688x_3x_5 + 2240x_3x_6 + 1920x_3x_7 + 1680x_3x_8 + 2016x_4x_5 + 1680x_4x_6 \\
&+1440x_4x_7 + 1260x_4x_8 + 1344x_5x_6 + 1152x_5x_7 + 1008x_5x_8 + 960x_6x_7 + 840x_6x_8 + 720x_7x_8 \\
&+40320x_1 + 20160x_2 + 13440x_3 + 10080x_4 + 8064x_5 + 6720x_6 + 5760x_7 + 5040x_8 + 40320 = \ldots
\end{aligned}
$$

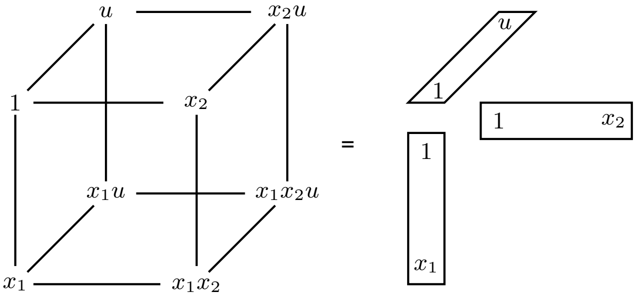
BUT QUITE SIMPLE FACTORIZATIONS MIGHT BE POSSIBLE

$$(1+x_1)(2+x_2)(3+x_3)(4+x_4)(5+x_5)(6+x_6)(7+x_7)(8+x_8)$$

WHEN IS THIS THE CASE?

MULTILINEAR MONOMIALS = RANK-1 TENSOR

Example: monomial tensor $\mathsf{M}(x_1, x_2, u)$ for a single input second order model



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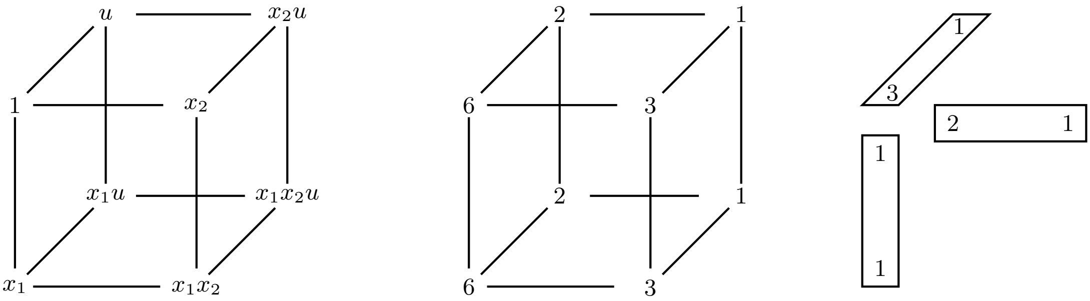
CANONICAL POLYADIC DECOMPOSITION (CPD) OF PARAMETER TENSOR POSSIBLE

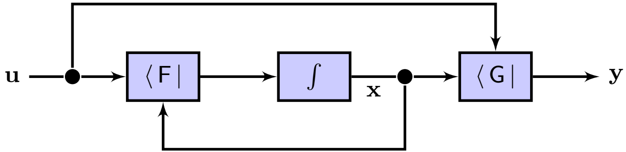


$$6 + 6x_1 + 3x_2 + 3x_1x_2 + 2u + 2x_1u + x_2u + x_1x_2u = (1 + x_1)(2 + x_2)(3 + u)$$

Complexity reduction of large models: not only sparsity but also low rank

MULTILINEAR STATE SPACE MODEL

Def.: input $\mathbf{u} \in \mathbb{R}^m$ (actors), output $\mathbf{y} \in \mathbb{R}^r$ (sensors), state $\mathbf{x} \in \mathbb{R}^n$ (storage)



Multilinear state space model (explicit) with $\mathsf{F} \in \mathbb{R}^{\times^{(n+m)} 2 \times n}$, $\mathsf{G} \in \mathbb{R}^{\times^{(n+m)} 2 \times r}$

$$\dot{\mathbf{x}} = \langle\, \mathsf{F} \mid \mathsf{M}(\mathbf{x}, \mathbf{u})\, \rangle$$
$$\mathbf{y} = \langle\, \mathsf{G} \mid \mathsf{M}(\mathbf{x}, \mathbf{u})\, \rangle$$

Monomial tensor $\mathsf{M}(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{2 \times 2 \times \cdots \times 2} := \mathbb{R}^{\times^{(n+m)} 2}$, contracted product $\langle \cdot \mid \cdot \rangle$

[Lichtenberg and Eichler 2011]

EXERCISES WITH THE MTI TOOLBOX [Lichtenberg et al. 2025]

A famous simple model

Lorenz Attractor

1. Type `lorenz` in MATLAB's Command Window, press `Start`
2. Type `edit lorenz` in MATLAB's Command Window
3. search for the definition of the model (at the end)
4.
```
function ydot = lorenzeq(y)
SIGMA = 10; RHO = 28; BETA = 8/3; % Parameter
A = [-BETA 0 y(2); 0 -SIGMA SIGMA; -y(2) RHO -1];
ydot = A*y;
```
5. Infer the nonlinear time-invariant (NTI) model

$$
\begin{aligned}
\dot{y}_1 &= -\beta y_1 + y_2 y_3 \\
\dot{y}_2 &= -\sigma y_2 + \sigma y_3 \\
\dot{y}_3 &= -y_1 y_2 + \rho y_2 - y_3
\end{aligned}
$$

EXERCISES WITH THE MTI TOOLBOX

Lorenz Attractor

$$\dot{x}_1 = -\beta x_1 + x_2 x_3$$
$$\dot{x}_2 = -\sigma x_2 + \sigma x_3$$
$$\dot{x}_3 = -x_1 x_2 + \rho x_2 - x_3$$

MTI modeling

1. Is the model multilinear ?
2. Which dimension has the transition tensor? $\mathsf{F} \in \mathbb{R}^{2\times2\times2\times3}$
3. Initialize this tensor as multidimensional array! `F = zeros(2,2,2,3);`
4. Set the nonzero values of `F`.

```
SIGMA = 10; RHO = 28; BETA = 8/3;
F(2,1,1,1)=-BETA ; F(1,2,2,1)=1;
F(1,2,1,2)=-SIGMA ; F(1,1,2,2)=SIGMA;
F(2,2,1,3)=-1 ; F(1,2,1,3)=RHO ; F(1,1,2,3)=-1;
```

5. Build a MTI model with this transition tensor ... `sys = mss(F);`

EXERCISES WITH THE MTI TOOLBOX

Lorenz Attractor

MTI simulation

1. Build a MTI model with this transition tensor ... `sys = mss(F);`
2. Run a simulation for 100 seconds from the initial state
3. `x0 = [20 5 -5];`
4. `y = msim(sys,[],100,x0);`
5. Plot the trajectory
6. `plot3(y(:,1),y(:,2),y(:,3));`

IMPLICIT MULTILINEAR TIME-INVARIANT (IMTI) MODELS

Explicit MTI state space model - reshaped

$$\dot{\mathbf{x}} - \langle \mathsf{F} \mid \mathsf{M}(\mathbf{x}, \mathbf{u}) \rangle = \mathbf{0}$$
$$\mathbf{y} - \langle \mathsf{G} \mid \mathsf{M}(\mathbf{x}, \mathbf{u}) \rangle = \mathbf{0}$$

Implicit MTI model [Lichtenberg et al. 2022]

$$\langle \mathsf{L} \mid \mathsf{M}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}, \mathbf{y}) \rangle = \mathbf{0}$$

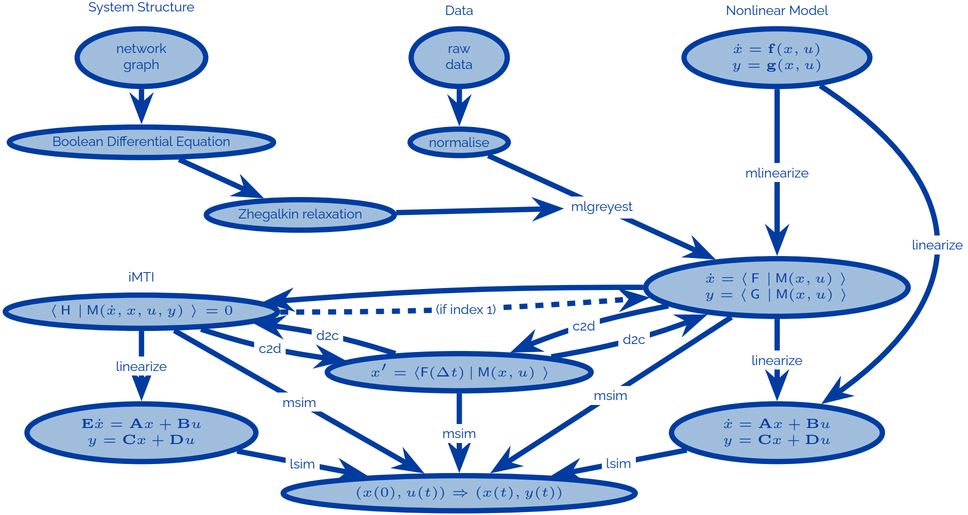
- iMTI model class is closed under composition
- differential-algebraic equations (DAEs)
- causal direction is lost
- other structural constraints like conservation laws can be included
- tensor decomposition methods enable large scale applications

LINEARIZATION & MULTILINEARIZATION

- Standard Linearization of nonlinear models around an operating POINT
- Using structured nonlinear models for power systems [Arevalo-Soler et al. 2025]
- Using efficient algorithms for MTI models [Kaufmann et al. 2023]

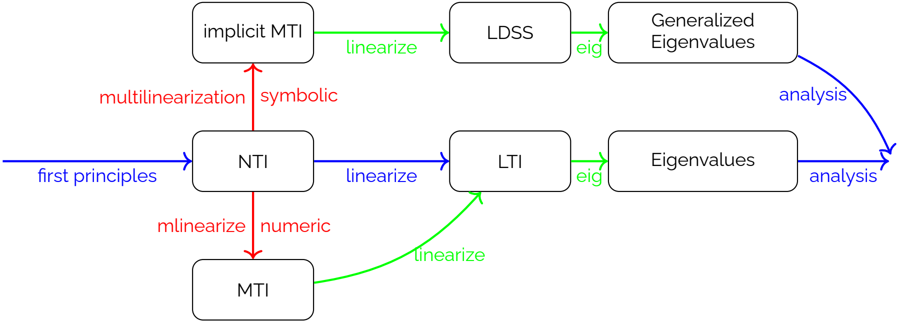
- Multilinearization of nonlinear models [Kruppa et al. 2014]
- Best approximation inside an operating REGION $\neq$ Taylor coeffs
- Improvements by sparse sampling techniques possible [Wong et al. 2025]
- Applications e.g. for complex dynamics of electrolysers [Luxa et al. 2025]

MTI TOOLBOX 2.1 FOR MATLAB



SMALL-SIGNAL POWER SYSTEM STABILITY ANALYSIS USING MULTILINEAR MODELS

Workflows - global picture



- STAMP tool presented by Marc Cheah Mañé , Dionysios Moutevelis, Onur Alican
- Below: explit MTI way with sparse grid methods presented by Teresa Wong next
- Above: implicit MTI way will be followed by Christoph Kaufmann finally

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NUMERICAL MULTILINEARIZATION OF NON-MULTILINEAR MODELS

$$\begin{aligned}\dot{\mathbf{x}} &= \langle \mathsf{F} | \mathsf{M}(\mathbf{x},\mathbf{u})\rangle \\ &= \mathbf{f}(\mathbf{x},\mathbf{u}) \\ &= \begin{bmatrix} f_1(\mathbf{x},\mathbf{u}) \\ \vdots \\ f_n(\mathbf{x},\mathbf{u})\end{bmatrix}\end{aligned}$$

Approximate nonlinear functions $f_i(\mathbf{x},\mathbf{u})$ with multilinear functions

$$\mathbf{h}(\mathbf{x},\mathbf{u}) \approx \mathbf{f}(\mathbf{x},\mathbf{u})$$

$$h_j(\mathbf{x},\mathbf{u}) = \sum_{i=1}^{\beta} \varphi_{\mathbf{i},j}\mu_{\mathbf{i}}(\mathbf{x},\mathbf{u})$$

e.g., when $(\mathbf{x},\mathbf{u}) = (x_1, x_2, u_1)$
$n = 2$, $m = 1$
$\mathbf{i} = [i_1, \dots i_8]^T$, $\beta := 2^{n+m} = 8$
$\forall j = 1, \dots, n = 1, 2$

$$\begin{aligned} h_j\,(\mathrm{x},\mathrm{u}) = {} & \varphi_{\mathrm{i}_1,j} + \varphi_{\mathrm{i}_2,j}x_1 + \varphi_{\mathrm{i}_3,j}x_2 + \varphi_{\mathrm{i}_4,j}x_1x_2 + \varphi_{\mathrm{i}_5,j}u_1 \\ & + \varphi_{\mathrm{i}_6,j}x_1u_1 + \varphi_{\mathrm{i}_7,j}x_2u_1 + \varphi_{\mathrm{i}_8,j}x_1x_2u_1 \end{aligned}$$

[Pangalos et al. 2013; Kruppa et al. 2014]

NUMERICAL MULTILINEARIZATION OF NON-MULTILINEAR MODELS

The approximation problem is to find a function h_j that minimizes the difference

$$\min_{h_j \in \mathcal{M}} \|h_j(\mathbf{x}, \mathbf{u}) - f_j(\mathbf{x}, \mathbf{u})\|_w, \ \forall j = 1, \dots, n$$

The norm of the inner product

$$\|f(\mathbf{x})\|_w := \sqrt{\langle f(\mathbf{x}), f(\mathbf{x}) \rangle_w}$$

$$\varphi_{\mathrm{i}_i, j} = \langle f_j\,(\mathrm{x}, \mathrm{u})\,,\ \mu_{\mathrm{i}_i}(\mathrm{x}, \mathrm{u}) \rangle$$

Inner product

$$\langle f(\mathbf{x}), g(\mathbf{x}) \rangle_w := \int_D f(\mathbf{x}) g(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}$$

Main task: Numerical integration using the Sparse Grids Matlab Kit [Piazzola and Tamellini 2023]

QUADRATURES

The sparse grids method is used to evaluate functions at each point of the grid and approximate an integral by quadrature formulas and interpolation.

A quadrature is the approximation of a definite integral as a weighted sum of function evaluations

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} f(x_i)\,w_i$$

QUADRATURES

$$\int_a^b f\left(x\right)dx \approx \sum_{i=1}^{N} f\left(x_i\right)w_i$$

$$Q_\ell := \sum_{i=1}^{m(\ell)} f(x_i)w_i \approx \int_a^b f(x)dx$$

x_i, w_i depend on the choice of quadrature rule

N - number of knots on the function $f(x_i)$, depends on the level-to-knots function $m(\ell)$, defined by the level of quadrature ℓ

USING THE SPARSE GRIDS METHOD IN NUMERICAL MULTILINEARIZATION

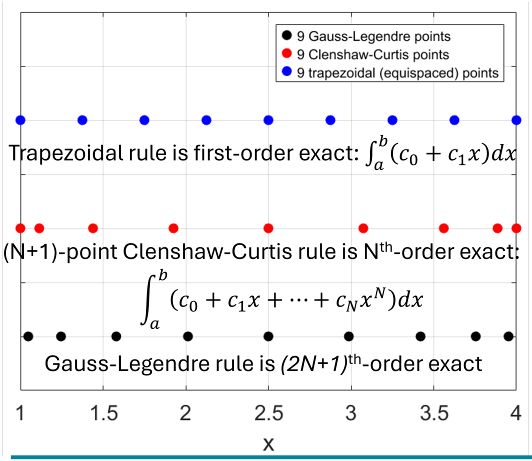
- Creating the sparse grid
- Evaluating functions for $\dot{x}$ and $\mu_{\mathbf{i}_i}$ using Simulink®
- Numerical integration on the sparse grid

MLINEARIZE IN MTI TOOLBOX

mlinearize(model, up_bnd, low_bnd, level, max_order)

- model: a model to be evaluated by feval
- up_bnd, low_bnd: upper and lower bounds of operating values for **states**, **inputs**, and **outputs**
- level: level of approximation in sparse grid
- max_order: limit on the no. of variables multiplied in a monomial
- Creates a MATLAB® class object “mss” to be included in the Simulink® simulation

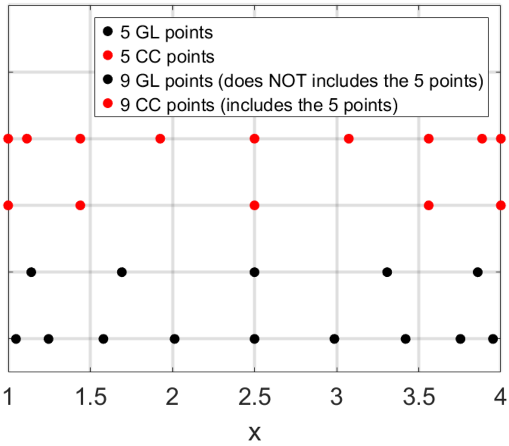
ACCURACY



- Polynomial exactness - the highest order polynomial that may be integrated exactly

- Note: some quadrature rules perform better for non-polynomial functions even if they have lower exactness

LEVEL-TO-KNOTS FUNCTION



- $m(\ell)$ determines the order of quadrature rule
- Examples of $m(\ell)$
  Linear function:

$$m(\ell) = \ell$$

Doubling function:

$$m(\ell) = \begin{cases} 1, & \text{if } \ell = 1, \\ 2^{\ell-1} + 1 & \text{if } \ell > 1 \end{cases}$$

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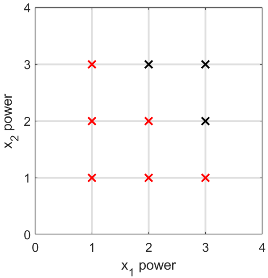
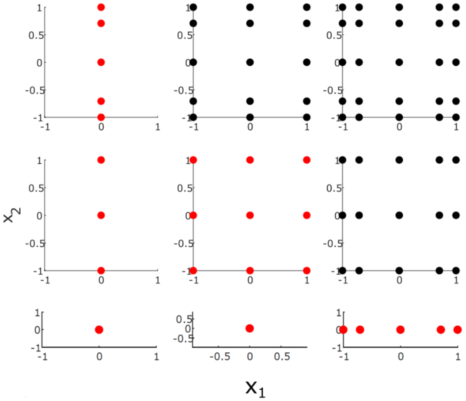
SMOLYAK QUADRATURE

$Q_3 \otimes Q_3$



Monomial: $x^{e_1} y^{e_2}$
Set limit: $e_1 + e_2 \leq 4$



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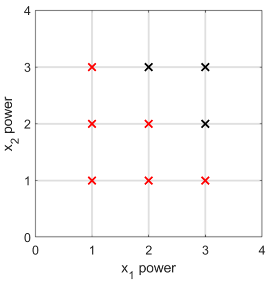
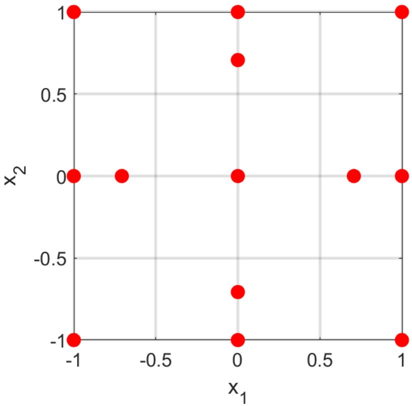
SMOLYAK QUADRATURE

$$Q_3 \otimes Q_3$$



Monomial: $x^{e_1} y^{e_2}$

Set limit: $e_1 + e_2 \leq 4$



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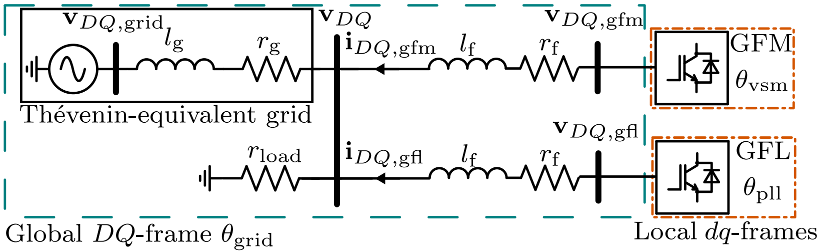
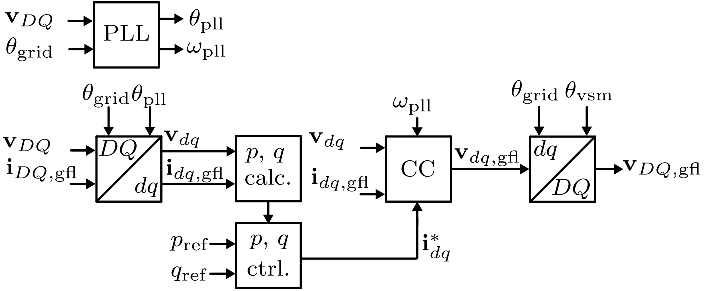
EXAMPLE: ESSENTIAL SYSTEM



EXAMPLE: ESSENTIAL SYSTEM

Grid-following Converter



NUMERICAL MULTILINEARIZATION OF THE GFL CONVERTER

Demo mlinearize

Exercise

- Run `open demo_mlinearize.mlx` in MATLAB's Command Window

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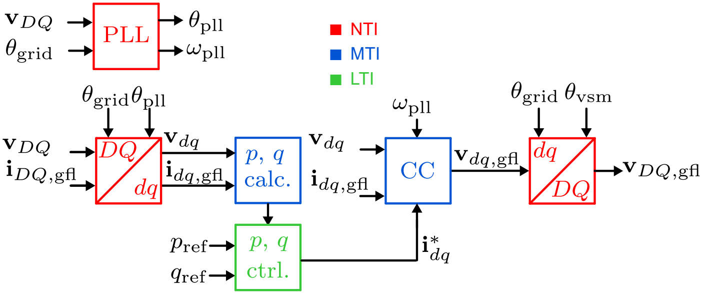
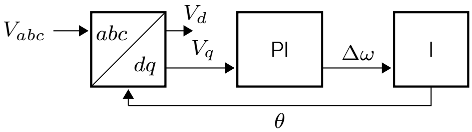
SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

Grid-following Converter



SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

SRF Phase-locked loop

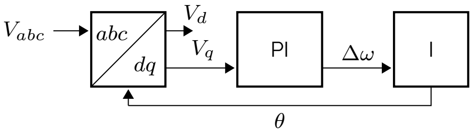


$$\begin{pmatrix} v_d \\ v_q \end{pmatrix} = \frac{2}{3} \begin{pmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin(\theta) & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \theta \\ x_I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ x_I \end{pmatrix} + \begin{pmatrix} 0 & k_P \\ 0 & k_I \end{pmatrix} \begin{pmatrix} v_\mathrm{d} \\ v_\mathrm{q} \end{pmatrix} + \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix}$$

SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

SRF Phase-locked loop



$$\dot{\theta} = x_{\mathrm{I}} + k_{\mathrm{p}} \frac{1}{3} \Big((-2v_{\mathrm{a}} + v_{\mathrm{b}} + v_{\mathrm{c}}) \sin(\theta) + \sqrt{3}\,(v_{\mathrm{b}} - v_{\mathrm{c}}) \cos(\theta) \Big) + \omega_0$$

$$\dot{x}_{\mathrm{I}} = k_{\mathrm{i}} \frac{1}{3} \Big((-2v_{\mathrm{a}} + v_{\mathrm{b}} + v_{\mathrm{c}}) \sin(\theta) + \sqrt{3}\,(v_{\mathrm{b}} - v_{\mathrm{c}}) \cos(\theta) \Big)$$

$$\dot{x}_1 = x_{\mathrm{I}} + k_{\mathrm{p}} \frac{1}{3} \Big((-2u_1 + u_2 + u_3) \sin(x_1) + \sqrt{3}\,(u_2 - u_3) \cos(x_1) \Big) + \omega_0$$

$$\dot{x}_2 = k_{\mathrm{i}} \frac{1}{3} \Big((-2u_1 + u_2 + u_3) \sin(x_1) + \sqrt{3}\,(u_2 - u_3) \cos(x_1) \Big)$$

SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

SRF Phase-locked loop

Motivated by [Savageau and Voit 1987], we define the following change of variables

$$z_1 = \cos(x_1) \quad z_2 = \sin(x_1) \quad z_3 = x_2$$

with the equality constraint is $0 = z_1^2 + z_2^2 - 1$.
This leads to [Kaufmann et al. 2025]

$$\dot{z}_1 = \dot{x}_1(-\sin(x_1)) = \left(z_3 + k_p \frac{1}{3}\left((-2u_1 + u_2 + u_3)\, z_2 + \sqrt{3}\,(u_2 - u_3)\, z_1\right) + \omega_0\right)(-z_2)$$

$$\dot{z}_2 = \dot{x}_1 \cos x_1 = \left(z_3 + k_p \frac{1}{3}\left((-2u_1 + u_2 + u_3)\, z_2 + \sqrt{3}\,(u_2 - u_3)\, z_1\right) + \omega_0\right) z_1$$

$$\dot{z}_3 = \dot{x}_2 = k_I \frac{1}{3}\left((-2u_1 + u_2 + u_3)\, z_2 + \sqrt{3}\,(u_2 - u_3)\, z_1\right)$$

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SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

SRF Phase-locked loop

$$
\begin{aligned}
0 &= \dot{z}_1 - \left(z_3 + k_p \frac{1}{3}\left(\left(-2u_1 + u_2 + u_3\right) z_2 + \sqrt{3}\left(u_2 - u_3\right) z_1\right) + \omega_0\right)\left(-z_5\right)\\
0 &= \dot{z}_2 - \left(z_3 + k_p \frac{1}{3}\left(\left(-2u_1 + u_2 + u_3\right) z_2 + \sqrt{3}\left(u_2 - u_3\right) z_1\right) + \omega_0\right) z_4\\
0 &= \dot{z}_3 - k_I \frac{1}{3}\left(\left(-2u_1 + u_2 + u_3\right) z_2 + \sqrt{3}\left(u_2 - u_3\right) z_1\right)\\
0 &= z_1 - z_4\\
0 &= z_2 - z_5
\end{aligned}
$$

$\rightarrow$ implicit multilinear time-invariant model of a PLL

[Kaufmann et al. 2025]

S. Casura[1,2], J. Fanals Batllori[3], G. Lichtenberg[1], T. Wong[1], C. Kaufmann[2], and C. Cateriano Yáñez[2]

SYMBOLIC MULTILINEARIZATION OF CONVERTER-RELEVANT NON-MULTILINEAR DYNAMICS

Demo PLL

Exercise

1. Run `open DemoPLL.mlx` in MATLAB's Command Window
2. Run the script and check the plot of time-domain simulation of the multilinear and the nonlinear model.

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LINEARIZATION OF NORM-1 MTI MODELS

Conceptual idea

$$f(\mathbf{x}) = (1 - |f_{x_1}| + f_{x_1} x_1)(1 - |f_{x_2}| + f_{x_2} x_2)(1 - |f_{x_3}| + f_{x_3} x_3) \dots$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \qquad f_{x_1} \qquad (1 - |f_{x_2}| + f_{x_2} \bar{x}_2)(1 - |f_{x_3}| + f_{x_3} \bar{x}_3) \dots$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = (1 - |f_{x_1}| + f_{x_1} \bar{x}_1) \qquad f_{x_2} \qquad (1 - |f_{x_3}| + f_{x_3} \bar{x}_3) \dots$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_3}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = (1 - |f_{x_1}| + f_{x_1} \bar{x}_1)(1 - |f_{x_2}| + f_{x_2} \bar{x}_2) \qquad f_{x_3} \qquad \dots$$

$$\vdots$$

[Kaufmann et al. 2023]

LINEARIZATION OF NORM-1 MTI MODELS

Conceptual idea

$$f(\mathbf{x}) = (1 - |f_{x_1}| + f_{x_1} x_1)(1 - |f_{x_2}| + f_{x_2} x_2)(1 - |f_{x_3}| + f_{x_3} x_3)\dots$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \qquad f_{x_1} \qquad (1 - |f_{x_2}| + f_{x_2} \bar{x}_2)(1 - |f_{x_3}| + f_{x_3} \bar{x}_3)\dots$$

How to compute the Jacobian efficiently for large higher order systems?

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_3}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = (1 - |f_{x_1}| + f_{x_1} \bar{x}_1)(1 - |f_{x_2}| + f_{x_2} \bar{x}_2) \qquad f_{x_3} \qquad \dots$$

$$\vdots$$

[Kaufmann et al. 2023]

LINEARIZATION OF NORM-1 MTI MODELS

Conceptual idea

- Saving computational cost by pre-computation $\mathbf{f}_\mathrm{p} \in \mathbb{R}^{1\times r}$

$$\mathbf{f}_\mathrm{p}(1) = (1 - |f_{x_1}| + f_{x_1}\bar{x}_1)(1 - |f_{x_2}| + f_{x_2}\bar{x}_2)(1 - |f_{x_3}| + f_{x_3}\bar{x}_3)\cdots$$

$$\begin{aligned}
\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} &= \frac{f_{x_1}}{(1 - |f_{x_1}| + f_{x_1}\bar{x}_1)} \cdot \mathbf{f}_\mathrm{p}(1) \\
&= \frac{f_{x_1}}{\cancel{(1 - |f_{x_1}| + f_{x_1}\bar{x}_1)}} \cdot \cancel{(1 - |f_{x_1}| + f_{x_1}\bar{x}_1)}(1 - |f_{x_2}| + f_{x_2}\bar{x}_2)(1 - |f_{x_3}| + f_{x_3}\bar{x}_3)\ldots \\
&= \frac{f_{x_1}}{(1 - |f_{x_1}| + f_{x_1}\bar{x}_1)} \cdot \mathbf{f}_\mathrm{p}(1) \\
\left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} &= \frac{f_{x_2}}{(1 - |f_{x_2}| + f_{x_2}\bar{x}_2)} \cdot \mathbf{f}_\mathrm{p}(1) \\
&\vdots \\
\left.\frac{\partial f(\mathbf{x})}{\partial x_n}\right|_{\mathbf{x}=\bar{\mathbf{x}}} &= \underbrace{\frac{f_{x_n}}{(1 - |f_{x_n}| + f_{x_n}\bar{x}_n)}}_{\mathbf{F}_\mathrm{d}\in\mathbb{R}^{n\times r}} \cdot \mathbf{f}_\mathrm{p}(r)
\end{aligned}$$

TenSyGrid

LINEARIZATION OF NORM-1 MTI MODELS

Conceptual idea

- Saving computational cost by pre-computation of $\mathbf{f}_\mathrm{p} \in \mathbb{R}^{1\times r}$ and $\mathbf{F}_\mathrm{d} \in \mathbb{R}^{n\times r}$

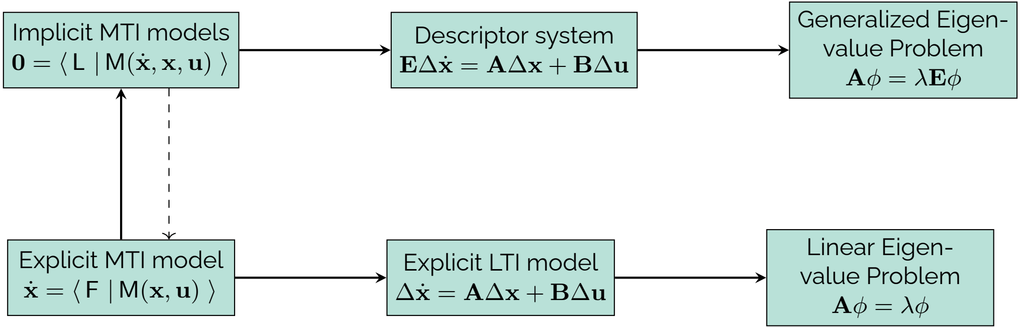
$$\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \mathbf{F}_\mathrm{d} \odot \mathbf{f}_\mathrm{p}$$

where $\odot$ is the element-wise multiplication of columns.

[Kaufmann et al. 2023]

LINEARIZATION OF IMTI MODELS AND GENERALIZED EIGENVALUES

Overview



[Kaufmann et al. 2023; Kaufmann et al. 2025]

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WORKFLOW POWER SYSTEM ANALYSIS USING THE MTI TOOLBOX

Overview

1. Select case study
2. Parametrize the system
3. Build iMTI model using component library
4. Run NTI model in Simulink
5. Initialize iMTI model using the operating point of the NTI model
6. Linearize iMTI model
7. Compute generalized eigenvalues

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EXERCISES WITH THE MTI TOOLBOX

Exercise

1. Run `open DemoSSA.mlx` in MATLAB's Command Window
2. Choose in **Case Study & Parametrization** the case `VoltageStep_SCR5_XR10_equalLoadDistribution_Rstep055` in the field `SimulationSetup.caseStudy`. Run the script until the section **Simulate Multilinear & Linear Descriptor Model** and compare the simulated response of the iMTI and NTI model
3. Now simulate the response of the linear descriptor model of the iMTI model by changing the field `SimulationSetup.Descriptor.simulate` to `yes`. Run the script **Simulate Multilinear &Linear Descriptor Model**. What do you see?
4. Change the study case to `VoltageStep_SCR5_XR10_equalLoadDistribution_Rstep` as before and compare the NTI, the iMTI and the LDSS response following the steps as before.
5. Check the eigenvalues for both cases by advancing to **Small-signal Analysis by Assessing the (Generalized) Eigenvalues**
6. Set `SimulationSetup.ComputationalAssessment` to `yes` and run the last section.

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